REPRESENTATION THEORY FINAL EXAM

This exam is of **75 marks** and is **4 hours long** - from 10 am to 2pm. Please **read all the questions carefully**. Please feel free to use whatever theorems you have learned in class after stating them clearly. You may also refer to the books by Steinberg or Fulton-Harris.

If you have any questions please call me at **+91 98804 59642** or email me at **rameshsreekantan@gmail.com**. Please sign the following statement and scan this sheet along with the rest.

I have not used any unfair or illegal means to answer any of the questions in this exam.

Name: **Signature:**

1. Let G be a group and V_1 and V_2 two representations. Define a representation of $G \times G$ on $V_1 \otimes V_2$ as follows:

$$(g_1, g_2) \cdot (v_1 \otimes v_2) = g_1 \cdot v_1 \otimes g_2 \cdot v_2$$

This is called the *external tensor product* and is denoted by $V_1 \boxtimes V_2$.

(1) What is the character of $V_1 \boxtimes V_2$? Justify your answer. 5

(2) Show that if V_1 and V_2 are irreducible so is $V_1 \boxtimes V_2$. 5

(3) Is the restriction to the diagonal subgroup $\Delta(G) = \{(g, g) | g \in G\} \subset G \times G$ irreducible? If 'yes' prove it. If 'no' prove it. If neither holds give examples to justify your answer. 10

2. Show that if G is a group of odd order it has an irreducible representation that cannot be defined over $\mathbb{R}$. 10

3. Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of Unit Quaternions - where i,j and k satisfy

$$i^2 = j^2 = k^2 = ijk = -1$$

(1) What are the possible dimensions of irreducible representations of Q? 5

(2) How many irreducible representations does Q have? 5

(3) What are the conjugacy classes of Q? 10

(4) Write down the character table of Q. 10

4. Let $G = PGL_2(\mathbb{F}_q) = GL_2(\mathbb{F}_q)/\mathbb{F}_q^*$ be the Projective General Linear Group of the finite field $\mathbb{F}_q$.

(1) How many elements are there in G? 5

(2) Using what you know about the character values of $GL_2(\mathbb{F}_q)$ write down the character table of $PGL_2(\mathbb{F}_2)$. 10

EXTRA QUESTION - just for your personal edification. Let G be a finite group. Show that if V is an irreducible representation of G of dimension > 1 with character χ_V then $\chi_V(g) = 0$ for some g in G. Hint: Show that the arithmetic mean of $|\chi(g)|$, $g \neq 1$ is < 1. Then use the relation between arithmetic mean and geometric mean to show that the geometric mean is < 1. Finally show that the geometric mean is in $\mathbb{Z}$ by arguing that it is an algebraic integer and is rational. Use that to conclude that it is 0.